

Lawson homology, morphic cohomology and Chow motives

Wenchuan Hu and Li Li

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Abstract

In this paper, the Lawson homology and morphic cohomology are defined on the Chow motives. We also define the rational coefficient Lawson homology and morphic cohomology of the Chow motives of finite quotient projective varieties. As a consequence, we obtain a formula for the Hilbert scheme of points on a smooth complex projective surface. Further discussion concerning generic finite maps is given. As a result, we give examples of self-product of smooth projective curves with nontrivial Griffiths groups by using a result of Ceresa.

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1 Introduction

The main purpose in this paper is to define the Lawson homology and morphic cohomology on the usual Chow motives as well as the Chow motives of finite quotient projective varieties.

The Lawson homology groups (resp. morphic cohomology groups) are the homotopy groups of the space of algebraic cycles (resp. algebraic cocycles), first studied by Friedlander and Lawson. We explain briefly their idea:

Let X be a complex projective variety and $\mathcal{Z}_p(X)$ be the abelian group of algebraic cycles of dimension p on X . There is a natural topology, namely Chow topology, on this abelian group which is independent of the projective embedding of X . The Lawson homology $L_p H_k(X)$ is defined to be the homotopy group

$$L_p H_k(X) := \begin{cases} \pi_{k-2p}(\mathcal{Z}_p(X)), & \text{if } k \geq 2p \\ 0, & \text{if } k < 2p \end{cases}$$

(cf. [F], [L1], [L2]). The topological group $\mathcal{Z}^p(X)$ of all algebraic cocycles of codimension- p on X is defined as a homotopy quotient completion (cf. [FL1], Definition 2.8)

$$\mathcal{Z}^q(X) := [\mathfrak{Mor}(X, \mathcal{C}_0(\mathbb{P}^q)) / \mathfrak{Mor}(X, \mathcal{C}_0(\mathbb{P}^{q-1}))]^+ = \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbb{A}^q)).$$

Take the $(l-2q)$ -th homotopy group of the space of algebraic cocycles instead of algebraic cycles, we get the morphic cohomology $L^q H^l(X)$. (Partial motivation to study the homotopy of the cycle space is Almgren's isomorphism, which asserts that for a topological space X satisfying reasonable conditions,

$$H_k(X) \cong \pi_{k-r}(Z_r(X)),$$

where $Z_r(X)$ is the space of r -dimensional integral cycles, i.e., integral currents without boundary.)

Now we fix our notation of Chow motives. Let \mathcal{V} denote the category of (not necessarily connected) complex smooth projective varieties. Given two smooth projective varieties X and Y . Suppose $X = \coprod X_\alpha$ is the decomposition of X into irreducible components. The group of correspondences of degree r from X to Y is defined as

$$Corr^r(X, Y) := \oplus Ch^{\dim X_\alpha + r}(X_\alpha \times Y),$$

moreover, its tensor with \mathbb{Q} is denoted by $Corr_{\mathbb{Q}}^r(X, Y)$. The composition of two correspondences $f \in Corr^r(X, Y)$ and $g \in Corr^s(Y, Z)$ gives a correspondence in $Corr^{r+s}(X, Z)$. A correspondence $\mathbf{p} \in Corr^0(X, X)$ is called a projector of X if $\mathbf{p}^2 = \mathbf{p}$. The category of Chow motives CHM is given as follows (cf. [CH] for the version we used here): Objects in CHM are triples (X, \mathbf{p}, r) , or denoted by $h(X, \mathbf{p})(-r)$, where $X \in \mathcal{V}$, \mathbf{p} is a projector of X , $r \in \mathbb{Z}$. In particular, the motive $h(X, id_X)(-r)$ is simply denoted by $h(X)(-r)$. Morphisms are defined as

$$Hom_{CHM}((X, \mathbf{p}, r), (Y, \mathbf{q}, s)) := \mathbf{q} \circ Corr^{s-r}(X, Y) \circ \mathbf{p}.$$

The composition of morphisms is defined as the composition of correspondences.

The following result states a relation of motives and the morphic cohomology. (Analogous result holds for Lawson homology, cf. Theorem 4.3 (i).)

Theorem 1.1 (Theorem 4.1) ¹ *Given any $q, l \in \mathbb{Z}$, the morphic cohomology $L^q H^l$ defines a covariant functor from the category CHM to the category of abelian groups as follows:*

$$L^q H^l(X, \mathbf{p}, r) := \mathbf{p}_*(L^{q+r} H^{l+2r}(X)) \subseteq L^{q+r} H^{l+2r}(X).$$

Given a morphism $\Gamma \in Hom_{CHM}((X, \mathbf{p}, r), (Y, \mathbf{q}, s))$, the morphism

$$L^q H^l(\Gamma) : L^q H^l(X, \mathbf{p}, r) \rightarrow L^q H^l(Y, \mathbf{q}, s)$$

is defined as the restriction of the map

$$\Gamma_* : L^{q+r} H^{l+2r}(X) \rightarrow L^{q+s} H^{l+2s}(Y).$$

The advantage of this theorem is that we can apply results on motives to morphic cohomology theory (and Lawson homology). Examples are: the projective bundle theorem (Corollary 5.1) which is firstly proved by Friedlander and Gabber in [FG], and the blowup formula for Lawson homology (Corollary 5.2), which is proved by the first author in [Hu], a result of Lima-Filho ([LF3]) for projective manifolds admitting cell-decompositions. By applying the above theorem to a result of N. A. Karpenko in [K], we get certain decomposition of Lawson homology and the morphic cohomology (Corollary 5.3).

Then we discuss finite quotients of smooth (quasi-)projective varieties.

Our first observation is a natural relation between the rational coefficient Lawson homology of a smooth quasi-projective variety and the one of its quotient.

¹This theorem is implicit, in a different formulation, in [NZ], since morphic cohomology gives an example of oriented cohomology theory. But for the completeness of the paper and the convenience of the readers, we still state it here.

Proposition 1.1 (Proposition 3.1) *Let $\pi : X \rightarrow X' := X/G$ denote the quotient map of a quasi-projective variety with a faithful action of a finite group G . Then there is a canonical isomorphism*

$$\pi_* : (L_p H_k(X, \mathbb{Q}))^G \cong L_p H_k(X', \mathbb{Q}), \quad \text{for any } p, k \in \mathbb{Z}.$$

and an isomorphism, when X is projective, as follows

$$\pi_! : (L^q H^l(X, \mathbb{Q}))^G \cong L^q H^l(X', \mathbb{Q}), \quad \text{for any } q, l \in \mathbb{Z}.$$

Remark 1.1 *Friedlander and Walker proved the proposition (in the proof of Theorem 5.5 in [FW]) under the assumption of the smoothness of the quotient $X' = X/G$.*

The following is our main result for quotient varieties of smooth projective varieties by a finite group action. (Analogous result holds for Lawson homology, as stated in Theorem 4.3 (ii).)

Theorem 1.2 (Theorem 4.2) *Given any $q, l \in \mathbb{Z}$, the \mathbb{Q} -coefficient morphic cohomology $L^q H^l(-, \mathbb{Q})$ defines a covariant functor from the category $CH\mathcal{M}'$ of Chow motives of quotient varieties to the category of abelian groups as follows:*

$$L^q H^l((X', \mathbf{p}, r), \mathbb{Q}) := \mathbf{p}_*(L^{q+r} H^{l+2r}(X', \mathbb{Q})) \subseteq L^{q+r} H^{l+2r}(X', \mathbb{Q}).$$

Given a morphism $\Gamma \in Hom_{CH\mathcal{M}'}((X', \mathbf{p}, r), (Y', \mathbf{q}, s))$, the morphism

$$L^q H^l(\Gamma, \mathbb{Q}) : L^q H^l((X', \mathbf{p}, r), \mathbb{Q}) \rightarrow L^q H^l((Y', \mathbf{q}, s), \mathbb{Q})$$

is defined as the restriction of the map

$$\Gamma_* : L^{q+r} H^{l+2r}(X', \mathbb{Q}) \rightarrow L^{q+s} H^{l+2s}(Y', \mathbb{Q}).$$

As an application, we give a decomposition of the Lawson homology and morphic cohomology for the Hilbert scheme $X^{[n]}$ of n points on a smooth complex projective surface X . It is well-known that $X^{[n]}$ is nonsingular (cf. [Fo]). Let $X^{(n)}$ be the n -th symmetric product of X and let $\pi : X^{[n]} \rightarrow X^{(n)}$ be the natural morphism, namely the Hilbert-Chow morphism. We denote by $\mathfrak{P}(n)$ the set of partitions of n . Any $\nu \in \mathfrak{P}(n)$ determined a quotient variety $X^{(\nu)}$ which is a product of symmetric products of X (for detailed meaning of notations appeared here, the reader is referred to §5.2 or [dCM]). Apply the above theorem to the motivic decomposition of $X^{[n]}$ proved by de Cataldo and Migliorini in [dCM], we obtain the following theorem:

Theorem 1.3 (Theorem 5.3) *Let X be a smooth complex projective surface. Then there is an isomorphism of Lawson homology groups for all $p, k \in \mathbb{Z}$:*

$$\bigoplus_{\nu \in \mathfrak{P}(n)} L_{p-n+l(\nu)} H_{k-2n+2l(\nu)}(X^{(\nu)}, \mathbb{Q}) \longrightarrow L_p H_k(X^{[n]}, \mathbb{Q})$$

and an isomorphism of morphic cohomology groups

$$\bigoplus_{\nu \in \mathfrak{P}(n)} L^{q-n+l(\nu)} H^{l-2n+2l(\nu)-d_\nu}(X^{(\nu)}, \mathbb{Q}) \longrightarrow L^q H^l(X^{[n]}, \mathbb{Q}).$$

To state the further consequences, we need to introduce some notations. The continuous homomorphism $\mathcal{Z}_p(X) \hookrightarrow Z_{2p}(X)$ induces the cycle class map

$$\Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X).$$

Define $L_p H_k(X)_{hom} := \ker \Phi_{p,k}$. The special case $L_p H_{2p}(X)_{hom}$ gives the Griffiths group of p -cycles $\text{Griff}_p(X) := \mathcal{Z}_p(X)_{hom} / \mathcal{Z}_p(X)_{alg}$. (The isomorphism $L_p H_{2p}(X)_{hom} \cong \text{Griff}_p(X)$ is shown by Friedlander [F].)

Then we give the following applications using the idea of motives and correspondences: a new proof of a result of the first author that the Lawson homology groups $L_1 H_k(-)_{hom}$ and $L_{n-2} H_k(-)_{hom}$ are birational invariants; Some properties of the Lawson homology groups of unirational threefolds and fourfolds; *examples of self-products of generic curves carrying nontrivial Griffiths groups*.

Proposition 1.2 (Proposition 6.4) *Let C be generic smooth projective curve of genus $g \geq 3$ and let $X = C^g$ be the g -copies of self products of C . Then $\text{Griff}_p(X) \otimes \mathbb{Q}$ are nontrivial for all $1 \leq p \leq g - 2$.*

The paper is organized as follows: §2 is a review of the Lawson homology groups, the duality between morphic cohomology and Lawson homology, and intersection theory. In §3 we discuss the \mathbb{Q} -coefficient Lawson homology and morphic cohomology group of a finite quotient variety, and the intersection theory in this setting. §4 contains the main results that the Lawson homology and morphic cohomology can be defined for the usual Chow motives and the Chow motives of finite quotient projective varieties. This is based on the fact that the action of correspondences on Lawson homology are functorial, which is covered in §4.1 and §4.2. As applications, in §5.1 the projective bundle theorem and blow-up formula for Lawson homology are reproved in a different way; the computation of the Lawson homology for a smooth projective variety with cell-decompositions is regained. §5.2 is an example on the finite quotient of projective variety: the \mathbb{Q} -coefficient Lawson homology/morphic cohomology of Hilbert schemes of points on a smooth surface. §6 gives further results and applications concerning generic rational maps, some new examples with nontrivial Griffiths groups are built from known case.

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2 Lawson homology and morphic cohomology

In this section, we first review the definition of the Lawson homology $L_p H_k(X)$ for all integer p and explain that the properties of the original Lawson homology still hold.² Next, we will review morphic cohomology and the duality.

2.1 Lawson homology

Denote by $H^{-1}\mathbf{Abtop}$ the category of abelian topological groups in which homotopy equivalences are inverted.

Given a projective variety X , we denote by $\mathcal{Z}_p(X)$ ($p \geq 0$) the space of algebraic p -cycles on X with the natural Chow topology. When X is quasi-projective, Lima-Filho gave the definition of $\mathcal{Z}_p(X)$ as the quotient

$$\mathcal{Z}_p(X) := \mathcal{Z}_p(\overline{X}) / \mathcal{Z}_p(\overline{X} - X),$$

where \overline{X} is any projective closure of X (cf. [LF] and [FG]). He shows that $\mathcal{Z}_p(X)$ is well-defined up to isomorphism in the category $H^{-1}\mathbf{Abtop}$. As a consequence, the homotopy groups of $\mathcal{Z}_p(X)$ are independent of the choice of the projective closure \overline{X} .

Based on the homotopy property of Lawson homology ([FG], Prop.2.3), the Lawson homology groups can be defined for any integer p as follows, where \mathbb{A}^r denotes the affine space of dimension r :

Definition 2.1 *Let X be a quasi-projective variety. For a (possibly negative) integer p , define the cycle space $\mathcal{Z}_p(X)$ to be the homotopy equivalent class of $\mathcal{Z}_{p+r}(X \times \mathbb{A}^r)$ for any integer $r \geq \max(0, -p)$. (The homotopy property of Lawson homology guarantees that $\mathcal{Z}_p(X)$ is independent of the choice of r .)*

The Lawson homology group $L_p H_k(X)$ is defined as

$$L_p H_k(X) := \begin{cases} \pi_{k-2p}(\mathcal{Z}_p(X)), & \text{if } k \geq 2p; \\ 0, & \text{if } k < 2p. \end{cases}$$

Remark 2.1 *For $p \geq 0$, the above definition coincides with the original definition of Lawson homology groups. For $p < 0$, we have $L_p H_k(X) = \pi_{k-2p}(\mathcal{Z}_0(X \times \mathbb{A}^{-p})) = H_{k-2p}^{BM}(X \times \mathbb{A}^{-p}) = H_k^{BM}(X) = L_0 H_k(X)$ (cf. [FW]).*

Thus defined Lawson homology groups have expected functorial properties.

Definition 2.2 *Let $f : X \rightarrow Y$ be a proper morphism between two quasi-projective varieties. For $p \in \mathbb{Z}$, define the push-forward map*

$$f_* : \mathcal{Z}_p(X) \rightarrow \mathcal{Z}_p(Y)$$

²Friedlander pointed out to us that the consideration for $p < 0$ is implicit in the work of Barry Mazur and himself, and the formalism is worked out in the unpublished part of the thesis of Mircea Voineagu.

to be the one induced by

$$(f \times \text{id})_* : \mathcal{Z}_{p+r}(X \times \mathbb{A}^r) \rightarrow \mathcal{Z}_{p+r}(Y \times \mathbb{A}^r)$$

for a non-negative integer $r \geq -p$.

The following is essentially due to Frelander ([F], Prop.2.9).

Proposition 2.1 *The above definition of f_* does not depend on the choice of r . Moreover, for two proper morphisms of quasi-projective varieties $f : X \rightarrow Y$, $g : Y \rightarrow Z$, the following functoriality holds:*

$$(gf)_* \stackrel{h.e.}{\simeq} g_* f_* : \mathcal{Z}_p(X) \rightarrow \mathcal{Z}_p(Z), \quad \forall p \in \mathbb{Z}.$$

As a consequence,

$$(gf)_* = g_* f_* : L_p H_k(X) \rightarrow L_p H_k(Z), \quad \forall p, k \in \mathbb{Z}.$$

Similarly, the definition of the pull-back map can be extended to include the cycles of negative dimensions:

Definition 2.3 *Let X and Y be quasi-projective varieties. Let $f : X \rightarrow Y$ be a l.c.i. (local complete intersection) morphism of codimension d (i.e. f factors into a regular imbedding followed by a smooth morphism, and $\dim Y - \dim X = d$). For $p \in \mathbb{Z}$, define the pull-back map in the category $H^{-1}\mathbf{Abtop}$:*

$$f^* : \mathcal{Z}_p(Y) \rightarrow \mathcal{Z}_{p-d}(X)$$

to be the one induced by

$$f^* : \mathcal{Z}_{p+r}(Y \times \mathbb{A}^r) \rightarrow \mathcal{Z}_{p+r-d}(X \times \mathbb{A}^r)$$

for an integer $r \geq \max(0, -p, d - p)$.

Proposition 2.2 *The above definition of f^* does not depend on the choice of r . Moreover, let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two l.c.i. morphisms between quasi-projective varieties of codimension d and e respectively. Then the following functoriality holds in the category $H^{-1}\mathbf{Abtop}$:*

$$(gf)^* \stackrel{h.e.}{\simeq} f^* g^* : \mathcal{Z}_p(Z) \rightarrow \mathcal{Z}_{p-d-e}(X), \quad \forall p \in \mathbb{Z}.$$

As a consequence,

$$(gf)^* = f^* g^* : L_p H_k(Y) \rightarrow L_{p-d-e} H_{k-2d-2e}(X), \quad \forall p, k \in \mathbb{Z}.$$

Proof. For the first part, it is enough to show the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Z}_{p+r}(Y \times \mathbb{A}^r) & \xrightarrow{(f \times \text{id})^*} & \mathcal{Z}_{p+r-d}(X \times \mathbb{A}^r) \\ \pi^* \downarrow h.e. & & \pi^* \downarrow h.e. \\ \mathcal{Z}_{p+r+1}(Y \times \mathbb{A}^{r+1}) & \xrightarrow{(f \times \text{id})^*} & \mathcal{Z}_{p+r+1-d}(X \times \mathbb{A}^{r+1}) \end{array}$$

for f being a smooth morphism and for f being a regular imbedding, respectively. In the former case we can check that the diagram commutes by definition, the latter case is immediate from [FG] Theorem 3.4 (d).

To show the functoriality, choose an integer $r \geq \max(0, -p, e-p, d+e-p)$ and consider

$$\mathcal{Z}_{p+r}(Z \times \mathbb{A}^r) \xrightarrow{(f \times \text{id})^*} \mathcal{Z}_{p+r-e}(Y \times \mathbb{A}^r) \xrightarrow{(g \times \text{id})^*} \mathcal{Z}_{p+r-d-e}(X \times \mathbb{A}^r).$$

By the same method as in [Pe] Lemma 11c (also cf. proof of [Fu] Proposition 6.6(c)), we have $[(g \times \text{id}) \circ (f \times \text{id})]^* \xrightarrow{h.e.} (f \times \text{id})^*(g \times \text{id})^*$. The conclusion follows. \square

Remark 2.2 *If we define naively that $L_p H_k(X) = 0$ for $p < 0$, then the functoriality of pull-back maps does not hold in general. For example, let $X = Z = \mathbb{P}^1$ and let $Y = \text{pt}$ be a point. $f : X \rightarrow Y$ be the constant map and $g : Y \rightarrow Z$ maps Y to any point in Z . Then*

$$(g \circ f)^* : H_2(Z) = L_0 H_2(Z) \rightarrow L_0 H_2(X) = H_2(X)$$

maps the generator of $H_2(Z)$ to the generator of $H_2(X)$ by definition. On the other hand,

$$f^* g^* : L_0 H_2(Z) \rightarrow L_{-1} H_0(Y) \rightarrow L_0 H_2(X)$$

factor through $L_{-1} H_0(Y)$. Thus, to save the functoriality $((g \circ f)^ = f^* g^*)$, we have to define $L_p H_k(X)$ non-trivially.*

We would like to point out here that the statement on “ $\alpha_ = 0$ if $m < n - v$ ” in Lemma 12 in Peters’ paper [Pe] is imprecise. Despite of this minor imprecision, his statement $\alpha_* = 0$ on $L_m^{\text{hom}} H_l(X)$ if $m < n - v$ is still valid. (cf. Remark 6.1)*

2.2 Morp hic cohomology

The morphic cohomology is defined by Friedlander and Lawson [FL1, FL2] by considering the homotopy groups of algebraic cocycles.

Let $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ be the topological monoid of effective algebraic cocycles of relative dimension r with values in Y , which by definition is the abelian monoid of morphisms from X to the Chow monoid $\mathcal{C}_r(Y)$ provided with the compact open topology. When X is geometrically unbranched (e.g. when X is normal), $\mathfrak{Mor}(X, \mathcal{C}_r(Y))$ can be thought of as

the subset of effective cycles in $X \times Y$ of dimension $(r + \dim X)$ which is equidimensional over X .

The topological group $\mathcal{Z}^q(X)$ of all algebraic cocycles of codimension q on X is defined as a naive group completion (cf. [FL2] pg.538)

$$\mathcal{Z}^q(X) := [\mathfrak{Mor}(X, \mathcal{C}_0(\mathbb{P}^q)) / \mathfrak{Mor}(X, \mathcal{C}_0(\mathbb{P}^{q-1}))]^+ = \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbb{A}^q)).$$

Notice that the group $\mathcal{Z}^q(X)$ is not empty even when $q \geq \dim X$.

Definition 2.4 *Let X be a projective variety.³ The morphic cohomology groups are defined to be the homotopy groups of $\mathcal{Z}^q(X)$:*

$$L^q H^l(X) := \pi_{2q-l}(\mathcal{Z}^q(X)) \quad \text{if } 2q \geq l \text{ and } q \geq 0.$$

If $2q < l$ or $q < 0$, we define $L^q H^l(X) = 0$.

A duality map between morphic cohomology and Lawson homology is observed by Friedlander and Lawson ([FL2]). This duality can be generalized with minor changes to include the case of any indices.

Definition 2.5 *Let X be a projective variety of dimension m . The duality map*

$$\mathcal{D} : \mathcal{Z}^p(X) \rightarrow \mathcal{Z}_{m-p}(X), \quad \forall p \leq m$$

is defined by the graphing construction followed by the inverse of the natural homotopy equivalence $\mathcal{Z}_{m-p}(X) \simeq \mathcal{Z}_m(X \times \mathbb{A}^p)$ (which is meaningful even for $p > m$ by Definition 2.1).

Taking the homotopy groups, we get the induced map (also denoted by \mathcal{D} by abuse of notation)

$$\mathcal{D} : L^p H^k(X) \rightarrow L_{m-p} H_{2m-k}(X).$$

Now we recall some properties of the morphic cohomology groups and the duality map that are needed in the rest of the paper.

1. For any morphism $f : X \rightarrow Y$ between quasi-projective varieties, there is a pull-back morphism

$$f^* : L^p H^k(Y) \rightarrow L^p H^k(X).$$

When f has equidimensional fibers (e.g., a flat morphism or a finite morphism) between normal varieties, there are Gysin homomorphisms

$$f_! : L^p H^k(X) \rightarrow L^{p-c} H^{k-2c}(Y)$$

³We restrict ourselves to consider the morphic cohomology of the projective varieties only, thanks to the reminder of Friedlander that the formulation of morphic cohomology for quasi-projective varieties is quite delicate.

for $2p \geq k \geq 2c$, where $c = \dim X - \dim Y$ (implied by [FL1] Proposition 2.5).

When X and Y are smooth projective varieties, let $c = \dim X - \dim Y$ and define

$$f_* : L^p H^k(X) \rightarrow L^{p-c} H^{k-2c}(Y), \quad \forall p, k \in \mathbb{Z}$$

as $\mathcal{D}^{-1} f_* \mathcal{D}$, where \mathcal{D} is the duality between morphic cohomology and Lawson homology defined in Proposition 2.3 and f_* is the push-forward for Lawson homology.

Similarly, given $f' : X' \rightarrow Y'$ between finite quotient of smooth projective varieties, we define

$$f'_* := (\mathcal{D}')^{-1} f'_* \mathcal{D}' : L^p H^k(X', \mathbb{Q}) \rightarrow L^{p-c} H^{k-2c}(Y', \mathbb{Q})$$

where let $c = \dim X' - \dim Y'$, the push-forward map f'_* on the right hand side is the one for Lawson cohomology and $\mathcal{D}' = \mathcal{D} \otimes \mathbb{Q}$ is induced by the duality map \mathcal{D} (which is an isomorphism by Lemma 3.5).

2. There is a cup product

$$\# : L^p H^k(X) \otimes L^{p'} H^{k'}(X) \rightarrow L^{p+p'} H^{k+k'}(X)$$

natural with respect to morphisms ([FL1] Corollary 6.2), i.e. for $f : X \rightarrow Y$ a morphism between quasi-projective varieties,

$$f^*(\alpha \# \beta) = f^*(\alpha) \# f^*(\beta), \quad \forall \alpha, \beta \in L^* H^*(Y).$$

3. If X is smooth and projective, then the duality map

$$\mathcal{D} : L^p H^k(X) \rightarrow L_{m-p} H_{2m-k}(X), \quad \text{for } p \leq m$$

is an isomorphism compatible with the ring structures of $L^* H^*(X)$ and $L_* H_*(X)$, i.e.

$$\mathcal{D}(\alpha \# \beta) = D(\alpha) \bullet D(\beta), \quad \forall \alpha \in L^p H^k(X), \beta \in L^q H^l(X),$$

where $p, q \leq m$ and $p + q \leq m$. (This restriction on p, q is unnecessary, see Proposition 2.3 below.)

The duality behaves as expected for the Lawson homology with possibly negative dimension:

Proposition 2.3 (1) *Let $f : X \rightarrow Y$ be a morphism between projective varieties with dimension m and n , respectively. Then $f^* \mathcal{D} = \mathcal{D} f^*$. In another word, the following diagram commutes for any integer p :*

$$\begin{array}{ccc} L^p H^k(Y) & \xrightarrow{\mathcal{D}} & L_{n-p} H_{2n-k}(Y) \\ \downarrow f^* & & \downarrow f^* \\ L^p H^k(X) & \xrightarrow{\mathcal{D}} & L_{m-p} H_{2m-k}(X) \end{array}$$

(2) If X is smooth and projective, then the duality (which we call the Friedlander-Lawson duality)

$$\mathcal{D} : L^p H^k(X) \rightarrow L_{m-p} H_{2m-k}(X)$$

is a group isomorphism for any integer p .

(3) $\mathcal{D}(\alpha \# \beta) = D(\alpha) \bullet D(\beta), \forall \alpha, \beta \in L^* H^*(X)$. In another word, the duality map is compatible with the ring structures of morphic cohomology and of Lawson homology.

Proof. (1) is an immediate consequence of the known result (cf. [FL2] Proposition 2.2 and 2.3).

(2) follows directly from the proof of [FL2] Theorem 3.3.

(3) is proved in the next section §2.3. □

2.3 Intersection theory

In this section, assume X is a smooth quasi-projective variety. Then the diagonal map $\Delta : X \rightarrow X \times X$ is a regular imbedding. Let $\alpha \in L_p H_k(X), \beta \in L_q H_l(X)$. There is a natural map $\mathcal{Z}_p(X) \wedge \mathcal{Z}_q(X) \rightarrow \mathcal{Z}_{p+q}(X \times X)$, where “ \wedge ” is the smash product. Taking the homotopy groups at both sides, we get a natural map $L_p H_k(X) \times L_q H_l(X) \rightarrow L_{p+q} H_{k+l}(X \times X)$, and we denote the image of (α, β) under this map by $\alpha \times \beta$.

Definition 2.6 Let X be a smooth quasi-projective variety and $\Delta : X \rightarrow X \times X$ be the diagonal map. For any $\alpha \in L_p H_k(X), \beta \in L_q H_l(X)$, the intersection $\alpha \bullet \beta \in L_{p+q-m} H_{k+l-2m}(X)$ is defined as

$$\alpha \bullet \beta := \Delta^*(\alpha \times \beta).$$

Notice that in the above definition, no restriction is put on p, q, k, l and fortunately, the compatibility with pull-back $f^*(\alpha \bullet \beta) = f^* \alpha \bullet f^* \beta$, the compatibility with duality $\mathcal{D}(\alpha \# \beta) = D(\alpha) \bullet D(\beta)$ and the projection formula $f_*(f^* \alpha \bullet \beta) = \alpha \bullet f_* \beta$ still hold in this more general situation where the cycles of negative dimensions are allowed. The proof are essentially the same as the canonical case. We explain as follows.

First we prove the compatibility with duality:

Proof. (of Proposition 2.3 (3): $\mathcal{D}(\alpha \# \beta) = D(\alpha) \bullet D(\beta)$.) The proof is exactly the same as [FL2] Proposition 2.7 and its remark, where cycle spaces of negative dimensions, if appear, are understood as in Definition 2.1. □

The next proposition (1)(2)(3) is adapted from [Pe] Lemma 11, with minor revises, while (4) is adapted from [FG] Theorem 3.5 (b).

Proposition 2.4 Let X, Y, X', Y' be smooth quasi-projective varieties.

(1) Let $f : X \rightarrow Y$ be a morphism. Then

$$f^*(\alpha \bullet \beta) = f^* \alpha \bullet f^* \beta, \quad \forall \alpha \in L_p H_k(Y), \beta \in L_q H_l(Y).$$

(2) Suppose the following is a fibre square:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Then with $d = \dim Y - \dim X$, for any integers p, k , one has

$$f^* g_* = g'_* f'^* : L_p H_k(Y') \rightarrow L_{p-d} H_{k-2d}(X).$$

(3) Let $f : X \rightarrow Y$ be a morphism. Then the projection formula holds in for any integers p, q, k, l :

$$f_*(\alpha \bullet f^* \beta) = f_* \alpha \bullet \beta, \quad \alpha \in L_p H_k(X), \beta \in L_q H_l(Y).$$

(4) The intersection is graded-commutative and associative, i.e., $\forall \alpha \in L_p H_k(X), \beta \in L_q H_l(X), \gamma \in L_r H_m(X)$, we have

$$\begin{aligned} \alpha \bullet \beta &= (-1)^{kl} \beta \bullet \alpha, \\ (\alpha \bullet \beta) \bullet \gamma &= \alpha \bullet (\beta \bullet \gamma). \end{aligned}$$

Proof. (1) It is a immediate consequence of functoriality of pull-back (Proposition 2.2). Indeed, since any morphism between smooth quasi-projective varieties are l.c.i., we have

$$f^*(\alpha \bullet \beta) = f^* \Delta_X^*(\alpha \times \beta) = \Delta_Y^*(f \times f)^*(\alpha \times \beta) = f^* \alpha \bullet f^* \beta.$$

(2) Take an integer $r \geq \max(0, -p, -p + d)$ and consider the following diagram in the category $H^{-1} \mathfrak{Abtop}$:

$$\begin{array}{ccc} \mathcal{Z}_{p-d+r}(X' \times \mathbb{A}^r) & \xleftarrow{(f' \times \text{id})^*} & \mathcal{Z}_{p+r}(Y' \times \mathbb{A}^r) \\ \downarrow (g' \times \text{id})_* & & \downarrow (g \times \text{id})_* \\ \mathcal{Z}_{p-d+r}(X \times \mathbb{A}^r) & \xleftarrow{(f \times \text{id})^*} & \mathcal{Z}_{p+r}(Y \times \mathbb{A}^r) \end{array}$$

It commutes, by consider the case when f is a regular imbedding ([FG] Theorem 3.4 d) and the case when f is a flat morphism ([Fu] Proposition 1.7). Then by our definition of cycle spaces (Definition 2.1), the conclusion follows.

(3) Peters' proof is valid in this setting: let $\gamma_f = (\text{id}, f) = (\text{id} \times f) \circ \Delta_X : X \rightarrow X \times Y$. Consider the fibre square

$$\begin{array}{ccc} X & \xrightarrow{\gamma_f} & X \times Y \\ \downarrow f & & \downarrow f \times \text{id} \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array}$$

Then $f_* \alpha \bullet \beta = \Delta_Y^*(f_* \beta \times \beta) = \Delta_Y^*(f \times \text{id})_*(\alpha \times \beta) = f_*((\gamma_f)^*(\alpha \times \beta)) = f_*(\Delta_X^*(\text{id} \times f)^*(\alpha \times \beta)) = f_*(\Delta_X^*(\alpha \times f^* \beta)) = f_*(\alpha \bullet f^* \beta)$, where the third equality is by applying (2) to the above fibre square, the fourth is by the functoriality of pull-back (Proposition 2.2).

(4) The standard argument still applies for negative cycle spaces. \square

3 Quotient variety by a finite group

In this section we explore the relation between the Lawson homology of a quasi-projective variety and the Lawson homology of its finite quotient. The goal is to establish Proposition 3.1.

Suppose a finite group G acts faithfully on a quasi-projective variety X (that is, the only element in G fixing every point in X is the identity). The quotient $X' = X/G$ is again a quasi-projective variety (cf. [Hj] §10). Let $\pi : X \rightarrow X'$ denote the quotient map. We give the definition of pull-back π^* of algebraic cycles as in [Fu] Example 1.7.6:

For any subvariety W of X , let

$$I_W = \{g \in G : g|_W = \text{id}_W\}$$

be the inertia group. Let $e_W = \text{card}(I_W)$ be the order of the group I_W .

Definition 3.1 *A group homomorphism $\pi^* : \mathcal{Z}_p(X') \rightarrow \mathcal{Z}_p(X)$ for $p \geq 0$ is defined as follows: for any subvariety V of X' , let*

$$\pi^*[V] = \sum e_W [W],$$

the sum over all irreducible components of $\pi^{-1}(V)$.

In general, for a possibly negative p we take $r \geq \max(0, -p)$ and define $\pi^ : \mathcal{Z}_{p+r}(X' \times \mathbb{A}^r) \rightarrow \mathcal{Z}_{p+r}(X \times \mathbb{A}^r)$ same as above, which induces $\pi^* : \mathcal{Z}_p(X') \rightarrow \mathcal{Z}_p(X)$.*

Remark 3.1 *This definition is uniquely characterized by the fact that*

$$\pi^* \pi_* [W] = G[W] := \sum_{g \in G} g_* [W].$$

To induce from π^* a map between Lawson homology groups of X' and X , it is necessary to verify the continuity of π^* .

Lemma 3.1 *Let $\pi : X \rightarrow X' := X/G$ denote the quotient map of a quasi-projective variety with a faithful action of a finite group G . The map $\pi^* : \mathcal{Z}_p(X') \rightarrow \mathcal{Z}_p(X)$ is continuous. As a consequence, it induces a morphism $\pi^* : L_p H_k(X') \rightarrow L_p H_k(X)$.*

Proof. Without loss of generality, we can assume $p \geq 0$, since the case when $p < 0$ can be easily deduced from the case $p = 0$.

By [LF2] Theorem 3.1 and Theorem 5.8, for a complex algebraic variety (in particular, a complex quasi-projective variety) X , there are three equivalent definitions for the topology of $\mathcal{Z}_p(X)$, namely, the flat topology $\mathcal{Z}_p(X)^{fl}$, the equidimensional topology $\mathcal{Z}_p(X)^{eq}$, and $\mathcal{Z}_p(X)^{ch}$ defined via Chow varieties (which is the original definition of the topology of $\mathcal{Z}_p(X)$). Therefore it suffices to show the continuity for

$$\pi^* : \mathcal{Z}_p(X')^{fl} \rightarrow \mathcal{Z}_p(X)^{eq}.$$

Let S be a smooth projective variety. Given a cycle Γ' on $S \times X'$ which is flat over S and of relative dimension p . Consider $\Gamma := (\text{id} \times \pi)^*(\Gamma')$ (which is well defined since $\text{id} \times \pi$ is a finite quotient morphism). Notice that Γ may not be flat over S , but is still equidimensional over S of relative dimension p . Given $s \in S$, let $[\Gamma_s]$ be the intersection theoretic fiber over s . Then it suffices to show that $\pi^*([\Gamma'_s]) = [\Gamma_s]$ for any $s \in S$.

Notice that $(\text{id} \times \pi)_*[\Gamma] = (\text{id} \times \pi)_*(\text{id} \times \pi)^*[\Gamma'] = |G|[\Gamma']$. Then

$$\begin{aligned} \pi_*[\Gamma_s] &= (\text{id} \times \pi)_*([\Gamma] \cdot [s \times X]) \\ &= (\text{id} \times \pi)_*([\Gamma] \cdot (\text{id} \times \pi)^*[s \times X']) \\ &= (\text{id} \times \pi)_*[\Gamma] \cdot [s \times X'] \\ &= |G|[\Gamma'_s]. \end{aligned}$$

The notation \cdot denotes the refined intersection. The third equality is because of the projection formula for refined intersection.

Therefore

$$|G|\pi^*([\Gamma'_s]) = \pi^* (|G|[\Gamma'_s]) = \pi^*\pi_*[\Gamma_s] = G[\Gamma_s] = |G|[\Gamma_s],$$

where the last equality is by the invariance of Γ_s under the action of G .

Since π^* is a morphism of free abelian groups, so by dividing $|G|$ from both sides of the above equalities we conclude that

$$\pi^*(\Gamma'_s) = \Gamma_s,$$

which completes the proof. \square

We need the following elementary fact about homotopy groups of topological abelian groups.

Lemma 3.2 *Let $f_1, f_2 : Z_1 \rightarrow Z_2$ be two continuous homomorphisms between topological abelian groups. Then the induced homomorphism of the sum is the sum of the induced homomorphisms on the homotopy groups, i.e., $(f_1 + f_2)_* = (f_1)_* + (f_2)_* : \pi_k(Z_1) \rightarrow \pi_k(Z_2)$, for any integer $k \geq 0$.*

Proof. Let $\alpha \in \pi_k(Z_1)$ and $g \in \alpha$. Sometimes we also write $[g] = \alpha$. Then $(f_1 + f_2)_*(\alpha) = (f_1 + f_2)_*([g]) = [(f_1 + f_2) \circ g] = [f_1 \circ g + f_2 \circ g] = [f_1 \circ g] + [f_2 \circ g] = (f_1)_*([g]) + (f_2)_*([g]) = (f_1)_*(\alpha) + (f_2)_*(\alpha)$. That is what we want to prove. \square

Now we show the following relation between the Lawson homology groups of X and its quotient $X' = X/G$.

Proposition 3.1 *Let $\pi : X \rightarrow X' := X/G$ denote the quotient map of a quasi-projective variety with a faithful action of a finite group G . Then there is a canonical isomorphism*

$$\pi_* : (L_p H_k(X, \mathbb{Q}))^G \cong L_p H_k(X', \mathbb{Q}), \quad \text{for any } p, k \in \mathbb{Z}.$$

and an isomorphism (if X is projective)

$$\pi_! : (L^q H^l(X, \mathbb{Q}))^G \cong L^q H^l(X', \mathbb{Q}), \quad \text{for any } q, l \in \mathbb{Z}.$$

Proof. We provide here the proof of the isomorphism of π_* , since the isomorphism of $\pi_!$ can be proved similarly.

Consider the push-forward map π_* and the pull-back π^* which is continuous by Lemma 3.1. It is easy to verify from the definition that, on the cycle spaces,

$$\pi_* \pi^* = |G| \cdot \text{id} : \mathcal{Z}_p(X') \rightarrow \mathcal{Z}_p(X')$$

and

$$\pi^* \pi_* = \sum_{g \in G} g_* : \mathcal{Z}_p(X) \rightarrow \mathcal{Z}_p(X).$$

Therefore, we have corresponding identities on Lawson homology groups, by a property of homotopy groups of topological abelian groups (Lemma 3.2):

$$\pi_* \pi^* = |G| \cdot \text{id} : L_p H_k(X') \rightarrow L_p H_k(X'),$$

and

$$\pi^* \pi_* = \sum_{g \in G} g_* : L_p H_k(X) \rightarrow L_p H_k(X).$$

Then the conclusion follows by the following simple fact about vector spaces (Lemma 3.3). \square

Lemma 3.3 *Let V_1, V_2 be two \mathbb{Q} -vector spaces acted by a finite group G . Suppose G acts trivially on V_2 and denote the G -invariant subspace of V_1 by V_1^G . Let $\phi : V_1 \rightarrow V_2$, $\psi : V_2 \rightarrow V_1$ be two equivariant linear maps of vector spaces (i.e. $\phi(gx) = \phi(x)$ and $g\psi(y) = \psi(y)$, $\forall g \in G, x \in V_1, y \in V_2$). If the following two conditions are satisfied,*

$$i) \quad \forall y \in V_2, \phi \circ \psi(y) = |G| \cdot y,$$

$$ii) \quad \forall x \in V_1, \psi \circ \phi(x) = Gx := \sum_{g \in G} gx,$$

then $\phi|_{V_1^G} : V_1^G \rightarrow V_2$ is an isomorphism, with inverse $\psi/|G|$.

Proof. The surjectivity of $\phi|_{V_1^G}$ is because of the surjectivity of $\phi \circ \psi = |G| \cdot \text{id}_{V_2}$. For injectivity, suppose $x \in V_1^G$ satisfying $\phi(x) = 0$. Since x is invariant under G -action, $0 = \psi \circ \phi(x) = Gx = |G| \cdot x$, which implies that $x = 0$. \square

Next, we define a natural intersection ring structure on the \mathbb{Q} -coefficient Lawson homology groups of a finite quotient of a smooth quasi-projective variety.

Definition 3.2 Let X be a smooth quasi-projective variety with a finite group G acting on it faithfully. Denote the quotient map by $\pi : X \rightarrow X'$. For any $\alpha \in L_p H_k(X', \mathbb{Q})$ and $\beta \in L_q H_l(X', \mathbb{Q})$, the intersection $\alpha \cdot \beta$ in $L_{p+q-m} H_{k+l-2m}(X', \mathbb{Q})$ is defined as

$$\alpha \cdot \beta := \frac{1}{|G|} \pi_* (\pi^* \alpha \bullet \pi^* \beta) \quad (1)$$

where π^* is defined in Definition 3.1, and \bullet is defined in Definition 2.6.

Proposition 3.2 Assume further that X' , hence X , is projective. Then the intersection product defined as above depends only on X' , not on the choice of X and G .

The proof is postponed to the end of this section. Our method is to compare the above intersection product with the cup product of the morphic cohomology.

Lemma 3.4 Use the notation as in the above Definition 3.2. Then for any $p, q, r, k, l, m \in \mathbb{Z}$, $\alpha \in L_p H_k(X', \mathbb{Q})$, $\beta \in L_q H_l(X', \mathbb{Q})$ and $\gamma \in L_r H_m(X, \mathbb{Q})$, we have

- (1) $\pi^*(\alpha \cdot \beta) = \pi^*(\alpha) \bullet \pi^*(\beta)$,
- (2) $\pi_*((\pi^* \alpha) \bullet \gamma) = \alpha \cdot \pi_*(\gamma)$.

Proof. (1) The definition of π^* and Proposition 2.4 imply that both sides are invariant under the G -action. Therefore it is enough to show the equality

$$\pi_* \pi^*(\alpha \cdot \beta) = \pi_*(\pi^*(\alpha) \bullet \pi^*(\beta)).$$

Since $\pi_* \pi^* = |G| \cdot \text{id}$, then by Definition 3.2 the above equality holds.

(2) The right hand side equals to

$$\frac{1}{|G|} \pi_* (\pi^* \alpha \bullet \pi^* \pi_* \gamma) = \frac{1}{|G|} \pi_* (\pi^* \alpha \bullet \sum_{g \in G} g^* \gamma) = \frac{1}{|G|} \sum_{g \in G} \pi_* g^* ((g^{-1})^* \pi^* \alpha \bullet \gamma).$$

Since $\pi^* \alpha$ is G -invariant, $(g^{-1})^* \pi^* \alpha = \pi^* \alpha$. Moreover, $\pi_* g^* = \pi_*$. Therefore the above equals to the left hand side $\pi_* ((\pi^* \alpha) \bullet \gamma)$. \square

Lemma 3.5 Let $\pi : X \rightarrow X' = X/G$ be a finite quotient map where G acts faithfully on a projective normal variety X , then the following diagrams commute (here we denote by \mathcal{D}' the duality map for X'):

$$\begin{array}{ccc} L^r H^k(X, \mathbb{Q}) & \xrightarrow{\mathcal{D}} & L_{m-r} H_{2m-k}(X, \mathbb{Q}) \\ \downarrow \pi^! & & \downarrow \pi_* \\ L^r H^k(X', \mathbb{Q}) & \xrightarrow{\mathcal{D}'} & L_{m-r} H_{2m-k}(X', \mathbb{Q}) \end{array} \quad (2)$$

$$\begin{array}{ccc}
L^r H^k(X, \mathbb{Q}) & \xrightarrow{\mathcal{D}} & L_{m-r} H_{2m-k}(X, \mathbb{Q}) \\
\pi^* \uparrow & & \uparrow \pi^* \\
L^r H^k(X', \mathbb{Q}) & \xrightarrow{\mathcal{D}'} & L_{m-r} H_{2m-k}(X', \mathbb{Q})
\end{array}$$

As a consequence, \mathcal{D}' is an isomorphism.

Proof. It is easy to check that both diagrams hold on the level of cocycles (in place of morphic cohomology) and cycles (in place of Lawson homology). Then by Proposition 3.1 and the fact that \mathcal{D} is an isomorphism, we know \mathcal{D}' is also an isomorphism. \square

Proposition 3.3 *Let $\pi : X \rightarrow X' = X/G$ be a finite quotient map where G acts faithfully on a smooth projective variety X . Then for any $\alpha, \beta \in L^* H^*(X', \mathbb{Q})$, the duality map $\mathcal{D}' : L^* H^*(X', \mathbb{Q}) \rightarrow L_* H_*(X', \mathbb{Q})$ satisfies*

$$\mathcal{D}'(\alpha \# \beta) = \mathcal{D}'(\alpha) \cdot \mathcal{D}'(\beta).$$

Proof. We have

$$\begin{aligned}
|G| \mathcal{D}'(\alpha) \cdot \mathcal{D}'(\beta) &= \pi_* [\pi^* \mathcal{D}'(\alpha) \bullet \pi^* \mathcal{D}'(\beta)] = \pi_* [\mathcal{D}(\pi^* \alpha) \bullet \mathcal{D}(\pi^* \beta)] \\
&= \pi_* \mathcal{D}(\pi^* \alpha \# \pi^* \beta) = \pi_* \mathcal{D} \pi^*(\alpha \# \beta) = \mathcal{D}' \pi_* \pi^*(\alpha \# \beta) = |G| \mathcal{D}'(\alpha \# \beta).
\end{aligned}$$

where the second and fifth equalities are because of Lemma 3.5, the third is from Proposition 2.3 (3), the fourth is because the pull-back π^* is compatible with the product of morphic cohomology. \square

Proof. (of Proposition 3.2) By the above proposition, it is enough to show that the duality map \mathcal{D}' is surjective, since then the product in the Lawson homology $L_* H_*(X', \mathbb{Q})$ is determined by the cup product $\#$ in the morphic cohomology $L^* H^*(X', \mathbb{Q})$.

On the other hand, by assumption X is smooth projective, then \mathcal{D} is an isomorphism. Lemma 3.2 asserts that π_* is an isomorphism. Then by diagram (2) we know $\mathcal{D}' \pi_! = \pi_* \mathcal{D}$ is surjective, it follows that \mathcal{D}' is surjective. \square

4 Correspondences and Motives

4.1 The action of Correspondences between smooth varieties

Let X and Y be smooth projective varieties. A **correspondence** Γ from X to Y is a cycle (or an equivalent class of cycles depending on the context) on $X \times Y$. We denote the group of correspondences of rational equivalence classes between varieties X and Y by

$$\text{Corr}_d(X, Y) := \text{Ch}_{\dim X + d}(X \times Y).$$

In general without assuming the varieties X, Y to be connected, we define

$$\text{Corr}_d(X, Y) := \oplus \text{Ch}_{\dim X_\alpha + d}(X_\alpha \times Y),$$

where $X = \coprod X_\alpha$ is the decomposition of connected components of X .

Recall ([Fu], Chapter 16) that a correspondence $\Gamma \in \text{Corr}_d(X, Y)$ acts on Chow groups as follows

$$\begin{aligned} \Gamma_* : \text{Ch}_p(X) &\rightarrow \text{Ch}_{p+d}(Y) \\ \Gamma_*(u) &= p_{2*}(p_1^*u \bullet \Gamma) \end{aligned}$$

where p_1 (resp. p_2) denote the projection from $X \times Y$ onto X (resp. Y) and \bullet is the intersection product on the Chow group of the smooth variety $X \times Y$.

Let X, Y, Z be smooth projective varieties. The composition of two correspondences $\Gamma_1 \in \text{Corr}_{d_1}(X, Y)$ and $\Gamma_2 \in \text{Corr}_{d_2}(Y, Z)$ is given by the formula

$$\Gamma_2 \circ \Gamma_1 = p_{13*}(p_{12}^*\Gamma_1 \cdot p_{23}^*\Gamma_2) \in \text{Corr}_{d_1+d_2}(X, Z)$$

where p_{ij} , $i, j = 1, 2, 3$ are the projection of $X \times Y \times Z$ on the product of its i th and j th factors.

Follow the idea of Peters [Pe], we define the analogous homomorphisms on the level of Lawson homology by the same formula. Notice that for any $\Gamma \in \text{Corr}_d(X, Y)$, by modulo algebraic equivalence instead of rational equivalence relation it determines an element in

$$\pi_0(\mathcal{Z}_{\dim X + d}(X \times Y)) = L_{\dim X + d}H_{2\dim X + 2d}(X \times Y)$$

which is again denoted by Γ by abuse of notation.

Definition 4.1 *Let X, Y be smooth projective varieties, $\Gamma \in \text{Corr}_d(X, Y)$. Then for any element $\alpha \in L_p H_k(X)$, the push-forward morphism is defined by*

$$\begin{aligned} \Gamma_* : L_p H_k(X) &\rightarrow L_{p+d} H_{k+2d}(Y) \\ \Gamma_*(\alpha) &= p_{2*}(p_1^*\alpha \bullet \Gamma). \end{aligned}$$

Now we show that the push-forward morphism defined as above is functorial.

Proposition 4.1 *Let X, Y, Z be smooth projective varieties, $\Gamma_1 \in \text{Corr}_d(X, Y)$ and $\Gamma_2 \in \text{Corr}_e(Y, Z)$. Then for any $u \in L_p H_k(X)$, we have*

$$(\Gamma_2 \circ \Gamma_1)_* u = \Gamma_{2*} \Gamma_{1*} u \in L_{p+d+e} H_{k+2d+2e}(Z).$$

Proof. The proof is by applying basic properties of push-forward and pull-back of Lawson homology groups which we list below for convenience:

1. Graded commutativity and associativity (Proposition 2.4 (4)).

2. Functoriality of push-forward and pull-back: $(fg)^* = g^*f^*$, $(fg)_* = f_*g_*$ (Proposition 2.1 and 2.2).
3. Projection formula: $f_*(\alpha \bullet f^*\beta) = f_*\alpha \bullet \beta$ (Proposition 2.4 (3)).
4. Pull-back compatible with the intersection product: $f^*(\alpha \bullet \beta) = f^*\alpha \bullet f^*\beta$ (Proposition 2.4 (1)).
5. Given a fiber square

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

where f, g are proper, and p, q are flat, then $p^*f_* = g_*q^*$ (Proposition 2.4 (2)).

Denote by p_{XY}^{XYZ} the projection from $X \times Y \times Z$ to $X \times Y$, and similarly for other projections.

$$\begin{aligned} (\Gamma_2 \circ \Gamma_1)_*u &= p_{Z*}^{XZ}((\Gamma_2 \circ \Gamma_1) \bullet p_X^{XZ*}u) \\ &= p_{Z*}^{XZ}(p_{XZ*}^{XYZ}(p_{XY}^{XYZ*}\Gamma_1 \bullet p_{YZ}^{XYZ*}\Gamma_2) \bullet p_X^{XZ*}u) \\ &\stackrel{3}{=} p_{Z*}^{XZ}p_{XZ*}^{XYZ}((p_{XY}^{XYZ*}\Gamma_1 \bullet p_{YZ}^{XYZ*}\Gamma_2) \bullet p_{XZ}^{XYZ*}p_X^{XZ*}u) \\ &\stackrel{2}{=} p_{Z*}^{XYZ}((p_{XY}^{XYZ*}\Gamma_1 \bullet p_{YZ}^{XYZ*}\Gamma_2) \bullet p_X^{XYZ*}u) \\ &\stackrel{1}{=} p_{Z*}^{XYZ}(p_{YZ}^{XYZ*}\Gamma_2 \bullet (p_{XY}^{XYZ*}\Gamma_1 \bullet p_X^{XYZ*}u)) \\ &\stackrel{2,4}{=} p_{Z*}^{YZ}p_{YZ*}^{XYZ}(p_{YZ}^{XYZ*}\Gamma_2 \bullet (p_{XY}^{XYZ*}(\Gamma_1 \bullet p_X^{XY*}u))) \\ &\stackrel{3}{=} p_{Z*}^{YZ}(\Gamma_2 \bullet p_{YZ*}^{XYZ}p_{XY}^{XYZ*}(\Gamma_1 \bullet p_X^{XY*}u)) \\ &\stackrel{5}{=} p_{Z*}^{YZ}(\Gamma_2 \bullet p_Y^{YZ*}p_Y^{XY}(\Gamma_1 \bullet p_X^{XY*}u)) \\ &= \Gamma_{2*}\Gamma_{1*}u. \end{aligned}$$

where the first and last equalities hold by the definition of push-forward for Lawson homology (Definition 4.1). For the second equality we use the [FG] Theorem 3.5 c, which asserts that for an intersection pairing $\mathcal{Z}_p(X) \times \mathcal{Z}_q(X) \xrightarrow{\bullet} \mathcal{Z}_{p+q-\dim X}(X)$, applying 0-th homotopy π_0 yields the usual intersection product on algebraic equivalence class, hence is compatible with the ring structure for Chow groups. \square

Denote $Corr^d(X, Y) := Ch^{\dim X + d}(X \times Y) = Ch_{\dim Y - d}(X \times Y)$. By the duality isomorphism \mathcal{D} between morphic cohomology and Lawson homology, the analogous functorial property for morphic cohomology immediately follows:

Proposition 4.2 *Let X, Y, Z be smooth projective varieties, $\Gamma_1 \in Corr^d(X, Y)$ and $\Gamma_2 \in Corr^e(Y, Z)$. Then for any $u \in L^q H^l(X)$,*

$$(\Gamma_2 \circ \Gamma_1)_*u = \Gamma_{2*}\Gamma_{1*}u \in L^{q+d+e} H^{l+2d+2e}(Z).$$

\square

4.2 The action of Correspondences between quotient varieties

In this subsection, we extend the action of correspondences to the category of finite quotients of nonsingular projective varieties. The definition is formally the same as Definition 4.1, with the intersection as defined in Definition 3.2.

Definition 4.2 *Let X', Y' be two finite quotient varieties and let $\Gamma' \in \text{Corr}_d(X', Y')_{\mathbb{Q}}$. The push-forward Γ'_* is defined by*

$$\begin{aligned}\Gamma'_* : L_p H_k(X', \mathbb{Q}) &\rightarrow L_{p+d} H_{k+2d}(Y', \mathbb{Q}) \\ \Gamma'_*(u) &= p_{2*}(p_1^* u \cdot \Gamma')\end{aligned}$$

Note that the above definition implicitly uses Proposition 3.2, i.e. the intersection product on a finite quotient variety is well-defined.

An important property for the push-forward action of a correspondence is the following functoriality.

Proposition 4.3 *Let X', Y', Z' be finite quotient varieties, $\Gamma'_1 \in \text{Corr}_d(X', Y')$ and $\Gamma'_2 \in \text{Corr}_e(Y', Z')$. Then for any $u \in L_p H_k(X', \mathbb{Q})$,*

$$(\Gamma'_2 \circ \Gamma'_1)_* u = \Gamma'_{2*} \Gamma'_{1*} u \in L_{p+d+e} H_{k+2d+2e}(Z', \mathbb{Q}).$$

Proof. Let $X' = X/G_1$, $Y' = Y/G_2$, $Z' = Z/G_3$. Denote the three quotient maps by $\pi_1 : X \rightarrow X'$, $\pi_2 : Y \rightarrow Y'$, $\pi_3 : Z \rightarrow Z'$. Define $\Gamma_1 := (\pi_1 \times \pi_2)^* \Gamma'_1$ and $\Gamma_2 := (\pi_2 \times \pi_3)^* \Gamma'_2$. Consider the following diagram (which looks like a prism with three square faces and two triangular faces), our goal is to prove the bottom triangle commutes on the level of \mathbb{Q} -coefficient Lawson homology groups.

$$\begin{array}{ccccc} X & & \xrightarrow{\frac{\Gamma_2 \circ \Gamma_1}{|\Gamma_2| |\Gamma_3|}} & & Z \\ & \searrow & & \nearrow & \\ & & Y & & \\ \pi_1 \downarrow & & \downarrow \pi_2 & & \downarrow \pi_3 \\ & \searrow & & \nearrow & \\ X' & & \xrightarrow{\Gamma'_2 \circ \Gamma'_1} & & Z' \\ & \searrow & & \nearrow & \\ & & Y' & & \end{array}$$

The upper triangle of the prism induces a commutative triangle in \mathbb{Q} -coefficient Lawson homology by Proposition 4.1. The three squares also induce commutative squares in

\mathbb{Q} -coefficient Lawson homology. Indeed, for any $u \in L_p H_k(X)$,

$$\begin{aligned}
\Gamma'_{1*} \pi_{1*} u &= p'_{2*} (p'^{*}_1 \pi_{1*} u \cdot \Gamma'_1) \\
&= p'_{2*} \frac{(\pi_1 \times \pi_2)^*}{|G_1||G_2|} ((\pi_1 \times \pi_2)^* p'^{*}_1 \pi_{1*} u \bullet (\pi_1 \times \pi_2)^* \Gamma'_1) \\
&= \frac{\pi_{2*} p_{2*}}{|G_1||G_2|} (p^*_1 \pi^*_1 \pi_{1*} u \bullet \Gamma_1) \\
&= \frac{\pi_{2*} p_{2*}}{|G_1||G_2|} (p^*_1 (\sum_{g \in G_1} g_* u) \bullet \Gamma_1)
\end{aligned}$$

where in the third equality we use the fact $(\pi_1 \times \pi_2)^* p'^{*}_1 = p^*_1 \pi^*_1$, which is valid even on the level of cycles therefore valid on the level of Lawson homology.

Next, notice that for any $g \in G_1$, the identity $p^*_1 g_* = (g \times 1)_* p^*_1$ is valid on the level of cycles therefore valid for Lawson homology. Moreover, Γ_1 is invariant under the action of the group $(G_1 \times 1)$. Therefore by projection formula

$$p_{2*} (p^*_1 g_* u \bullet \Gamma_1) = p_{2*} ((g \times 1)_* p^*_1 u \bullet \Gamma_1) = p_{2*} (g \times 1)_* (p^*_1 u \bullet \Gamma_1) = p_{2*} (p^*_1 u \bullet \Gamma_1).$$

Continue the above calculation of $\Gamma'_{1*} \pi_{1*} u$:

$$\Gamma'_{1*} \pi_{1*} u = \frac{\pi_{2*}}{|G_1||G_2|} (|G_1| p_{2*} p^*_1 u \bullet \Gamma_1) = \pi_{2*} \left(\frac{\Gamma_1}{|G_2|} \right)_* u.$$

Thus the left square commutes. The commutativity of the other two squares are similar, while in the proof we need fact that

$$\frac{\Gamma_2 \circ \Gamma_1}{|G_2||G_3|} = (\pi_1 \times \pi_3)^* \left(\frac{\Gamma'_2 \circ \Gamma'_1}{|G_3|} \right).$$

Finally, since four of the five sides of the above prism induce commutative diagrams and $\pi_{1*} : L_p H_k(X, \mathbb{Q}) \rightarrow L_p H_k(X', \mathbb{Q})$ is surjective, the triangle at the bottom must commute, i.e.,

$$(\Gamma'_2 \circ \Gamma'_1)_* = \Gamma'_{2*} \Gamma'_{1*}.$$

□

In the similar situation as the last subsection, by the duality isomorphism \mathcal{D}' , we have a corresponding result to Proposition 4.2 for the morphic cohomology follows from Proposition 4.3.

Proposition 4.4 *Let X', Y', Z' be finite quotient varieties, $\Gamma'_1 \in \text{Corr}^d(X', Y')$ and $\Gamma'_2 \in \text{Corr}^e(Y', Z')$. Then for any $u \in L^q H^l(X', \mathbb{Q})$,*

$$(\Gamma'_2 \circ \Gamma'_1)_* u = \Gamma'_{2*} \Gamma'_{1*} u \in L^{q+d+e} H^{l+2d+2e}(Z', \mathbb{Q}).$$

□

4.3 Motive, Lawson homology and morphic cohomology

In this subsection, we explain that the morphic cohomology gives a covariant functor from the category of Chow motives to the category of bi-graded abelian groups. Analogously, the \mathbb{Q} -coefficient morphic cohomology gives a covariant functor from the category of Chow motives for finite quotient varieties to the category of bi-graded \mathbb{Q} -vector spaces.

We have recalled the definition of Chow motives in §1 (Introduction). The theory of Chow motives can be extended to $CH\mathcal{M}'$, the Chow motives of the category of quotient varieties of smooth projective varieties by finite groups ([dBV]). To be more precise, let \mathcal{V}' be the category of (not necessarily connected) varieties of the type X/G with $X \in Ob\mathcal{V}$ with an action of a finite group G . The objects of $CH\mathcal{M}'$ are the same as the objects of \mathcal{V}' , and the morphisms are defined similarly as in $CH\mathcal{M}$. We again have a contravariant functor $h : \mathcal{V}' \rightarrow CH\mathcal{M}'$.

Theorem 4.1 *Given any $q, l \in \mathbb{Z}$, the morphic cohomology $L^q H^l$ defines a covariant functor from the category $CH\mathcal{M}$ to the category of abelian groups as follows:*

$$L^q H^l(X, \mathbf{p}, r) := \mathbf{p}_*(L^{q+r} H^{l+2r}(X)) \subseteq L^{q+r} H^{l+2r}(X).$$

Given a morphism $\Gamma \in Hom_{CH\mathcal{M}}((X, \mathbf{p}, r), (Y, \mathbf{q}, s))$, the morphism

$$L^q H^l(\Gamma) : L^q H^l(X, \mathbf{p}, r) \rightarrow L^q H^l(Y, \mathbf{q}, s)$$

is defined as the restriction of the map

$$\Gamma_* : L^{q+r} H^{l+2r}(X) \rightarrow L^{q+s} H^{l+2s}(Y).$$

Proof. First, we need to show that $L^q H^l(\Gamma)$ is well defined, i.e. the following diagram commutes,

$$\begin{array}{ccc} L^{q+r} H^{l+2r}(X) & \xrightarrow{\Gamma_*} & L^{q+s} H^{l+2s}(Y) \\ \downarrow \mathbf{p}_* & & \downarrow \mathbf{q}_* \\ L^{q+r} H^{l+2r}(X) & \xrightarrow{\Gamma_*} & L^{q+s} H^{l+2s}(Y) \end{array}$$

It commutes because of $\mathbf{q} \circ \Gamma = \Gamma \circ \mathbf{p}$ and Proposition 4.2.

Then we need to verify the functoriality of $L^q H^l$. This again follows from Proposition 4.2. \square

Remark 4.1 *For our purpose, the category of Chow motives $CH\mathcal{M}$ can be replaced by the category of algebraic motives $CH_A\mathcal{M}$ whose objects are the same as $CH\mathcal{M}$, while morphisms are defined to be*

$$Hom_{CH_A\mathcal{M}}((X, \mathbf{p}, r), (Y, \mathbf{q}, s)) := \mathbf{q} \circ Corr_{alg}^{s-r}(X, Y) \circ \mathbf{p}$$

where $Corr_{alg}^{s-r}(X, Y) = Corr^{s-r}(X, Y) / \{\text{algebraic equivalence}\}$.

Similarly, for the Chow motives of finite quotient varieties we have:

Theorem 4.2 *Given any $q, l \in \mathbb{Z}$, the \mathbb{Q} -coefficient morphic cohomology $L^q H^l(-, \mathbb{Q})$ defines a covariant functor from the category CHM' to the category of abelian groups as follows:*

$$L^q H^l((X', \mathbf{p}, r), \mathbb{Q}) := \mathbf{p}_*(L^{q+r} H^{l+2r}(X', \mathbb{Q})) \subseteq L^{q+r} H^{l+2r}(X', \mathbb{Q}).$$

Given a morphism $\Gamma \in Hom_{CHM'}((X', \mathbf{p}, r), (Y', \mathbf{q}, s))$, the morphism

$$L^q H^l(\Gamma, \mathbb{Q}) : L^q H^l((X', \mathbf{p}, r), \mathbb{Q}) \rightarrow L^q H^l((Y', \mathbf{q}, s), \mathbb{Q})$$

is defined as the restriction of map

$$\Gamma_* : L^{q+r} H^{l+2r}(X', \mathbb{Q}) \rightarrow L^{q+s} H^{l+2s}(Y', \mathbb{Q}).$$

Proof. Same as the proof of Theorem 4.1. Proposition 4.4 implies that $L^q H^l(-, \mathbb{Q})$ is well-defined and functorial. \square

There are corresponding versions of Theorem 4.1 and 4.2 for Lawson homology, with almost the same proof hence we skip it and only give the statement:

Theorem 4.3 *For any $p, k \in \mathbb{Z}$,*

(i) the Lawson homology $L_p H_k$ defines a contravariant functor from the category CHM to the category of abelian groups as follows:

$$L_p H_k(X, \mathbf{p}, r) := \mathbf{p}_*(L_{p+r} H_{k+2r}(X)) \subseteq L_{p+r} H_{k+2r}(X).$$

Given a morphism $\Gamma \in Hom_{CHM}((X, \mathbf{p}, r), (Y, \mathbf{q}, s))$, the morphism

$$L_p H_k(\Gamma) : L_p H_k(Y, \mathbf{q}, s) \rightarrow L_p H_k(X, \mathbf{p}, r)$$

is the restriction of map $({}^t\Gamma)_ : L_{p+s} H_{k+2s}(Y) \rightarrow L_{p+r} H_{k+2r}(X)$.*

(ii) the \mathbb{Q} -coefficient Lawson cohomology $L_p H_k(-, \mathbb{Q})$ defines a contravariant functor from the category CHM' to the category of \mathbb{Q} -vector spaces as follows:

$$L_p H_k((X', \mathbf{p}, r), \mathbb{Q}) := \mathbf{p}_*(L_{p+r} H_{k+2r}(X', \mathbb{Q})) \subseteq L_{p+r} H_{k+2r}(X', \mathbb{Q}).$$

Given a morphism $\Gamma \in Hom_{CHM'}((X', \mathbf{p}, r), (Y', \mathbf{q}, s))$, the morphism

$$L_p H_k(\Gamma, \mathbb{Q}) : L_p H_k((Y', \mathbf{q}, s), \mathbb{Q}) \rightarrow L_p H_k((X', \mathbf{p}, r), \mathbb{Q})$$

is the restriction of map $({}^t\Gamma)_ : L_{p+s} H_{k+2s}(Y', \mathbb{Q}) \rightarrow L_{p+r} H_{k+2r}(X', \mathbb{Q})$.*

5 Applications

5.1 Projective bundles, blow-ups, and cell-decomposition

As application of the connection between Lawson homology and the morphic cohomology, we reobtain formulas for projective bundles, blow-ups, and smooth varieties admitting a cell-decomposition. However, we require varieties to be smooth in these cases.

We start from the well known motivic decompositions for a projective bundle and for a blow-up. Let \mathbb{P} be a projective bundle over a smooth projective variety X with fiber \mathbb{P}^n . The following motivic decomposition is proved in [M],

$$h(\mathbb{P}) \simeq h(X) \oplus h(X)(1) \oplus \cdots \oplus h(X)(n).$$

then by Theorem 4.1 and Theorem 4.3 (recall that $(X, \text{id}_X, r) = h(X)(-r)$), we have the following result proved by Friedlander and Gabber:

Corollary 5.1 (Projective Bundle Theorem, [FG]) *Let \mathbb{P} be a projective bundle over a smooth projective variety X with fiber \mathbb{P}^n . Then the following decompositions hold for morphic cohomology and Lawson homology:*

$$\begin{aligned} L^q H^l(\mathbb{P}) &\simeq L^q H^l(X) \oplus L^{q-1} H^{l-2}(X) \oplus \cdots \oplus L^{q-n} H^{l-2n}(X), \quad \forall q, l \in \mathbb{Z}. \\ L_p H_k(\mathbb{P}) &\simeq L_p H_k(X) \oplus L_{p-1} H_{k-2}(X) \oplus \cdots \oplus L_{p-n} H_{k-2n}(X), \quad \forall p, k \in \mathbb{Z}. \end{aligned} \quad (3)$$

Let X be a smooth projective variety and $j_0 : V \hookrightarrow X$ a smooth subvariety of codimension $n \geq 2$. Let \tilde{X} be the blowup of X along V . Because of Theorem 4.1, Theorem 4.3 and the motivic decomposition (cf. [M])

$$h(\tilde{X}) \simeq h(X) \oplus h(V)(1) \oplus \cdots \oplus h(V)(n-1),$$

we get the blowup formula for the morphic cohomology and Lawson homology:

Corollary 5.2 ([Hu]) *Let \tilde{X} be the blow-up of a smooth projective variety X along a smooth subvariety V of codimension n . Then*

$$\begin{aligned} L^q H^l(\tilde{X}) &\simeq L^q H^l(X) \oplus \bigoplus_{i=1}^{n-1} L^{q-i} H^{l-2i}(V), \\ L_p H_k(\tilde{X}) &\simeq L_p H_k(X) \oplus \bigoplus_{i=1}^{n-1} L_{p-i} H_{k-2i}(V). \end{aligned}$$

More generally, recall the following result proved by N. A. Karpenko in [K]:

Theorem 5.1 (Karpenko) *Let X be a smooth projective variety. Assume X admits a filtration by closed subvarieties $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ such that there exist flat morphisms $f_i : X_i - X_{i-1} \rightarrow Y_i$, of relative dimension m_i over smooth projective varieties Y_i ($1 \leq i \leq n$), such that the fiber of every f_i over every point y of Y_i is isomorphic to the affine space \mathbb{C}^{m_i} . Then there exists an isomorphism in CHM*

$$h(X) \simeq \bigoplus_{i=1}^n h(Y_i)(m_i). \quad (4)$$

We immediately get the following

Corollary 5.3 *Using the notations in Theorem 5.1, we have*

$$L^q H^l(X) \simeq \bigoplus_{i=1}^n L^{q-m_i} H^{l-2m_i}(Y_i), \quad (5)$$

$$L_p H_k(X) \simeq \bigoplus_{i=1}^n L_{p-m_i} H_{k-2m_i}(Y_i). \quad (6)$$

In particular, the isomorphism (6) can be used to compute the Lawson homology for Grassmann bundles of projective vector bundles.

Proof. Note that

$$\begin{aligned} Id_X &\in Hom_{CHM}((X, id, 0), \bigoplus_{r=0}^n (Y_i, id, -m_i)) \\ &= \bigoplus_{i=0}^n Hom_{CHM}((X, id, 0), (Y_i, id, -m_i)) \\ &= \bigoplus_{i=0}^n Corr^{-m_i}(X, Y_i), \end{aligned} \quad (7)$$

hence $Id_X = \bigoplus_{i=0}^n \Gamma_i$, where $\Gamma_i \in Corr^{-m_i}(X, Y_i)$. By Theorem 4.1 and 5.1, we have Equation (5).

Similarly, by Theorem 4.3 and 5.1, we obtain Equation (6). \square

Remark 5.1 *It was pointed out to us by Friedlander that the decomposition of motives in Equation (4) implies the decomposition of any oriented cohomology theory (cf. [NZ]). Friedlander and Walker showed that the Lawson homology and morphic cohomology are such theories for varieties over \mathbb{R} and \mathbb{C} (cf. [FW2] and references therein by the same authors). Hence, Corollary 5.3 was implied from those, although explicit formula was not written down.*

Remark 5.2 *By the Friedlander-Lawson duality (cf. Proposition 2.3), the Equation (5) is equivalent to the following formula in terms of Lawson homology groups:*

$$L_p H_k(X) \simeq \bigoplus_{i=1}^n L_{p-d_i} H_{k-2d_i}(Y_i). \quad (8)$$

By comparing Equation (6) and (8), we obtained visible obstructions for a collection of pairs $\{(Y_i, m_i)\}_{i=1}^n$, where Y_i is a smooth projective variety and m_i is a positive integer for each $1 \leq i \leq n$, to be the decomposition of a smooth projective variety X in the sense of Theorem 5.1.

5.2 Hilbert scheme of points on a surface

In this section, we will compute the rational coefficient morphic cohomology and Lawson homology for Hilbert scheme of points on a smooth complex projective surface.

Let X be a smooth projective surface, let $X^{(n)}$ be its n -th symmetric product, let $X^{[n]}$ be the Hilbert scheme of 0-dimensional subschemes of X of length n , and let $\pi : X^{[n]} \rightarrow X^{(n)}$ be the Hilbert-Chow morphism. It is well-known that $X^{[n]}$ is nonsingular. We denote by $\mathfrak{P}(n)$ the set of partitions of n and $p(n)$ its cardinality. For any $\nu \in \mathfrak{P}(n)$, we denote by $l(\nu)$ its length, and define $X_\nu^{(n)}$ to be the locally closed subset of points in $X^{(n)}$ of the type $\nu_1 x_1 + \cdots + \nu_{l(\nu)} x_{l(\nu)}$, with $x_h \in X$ and $x_i \neq x_j$ for every $i \neq j$. Define $X_\nu^{[n]}$ to be the reduced scheme $(\pi^{-1}(X_\nu^{(n)}))_{red}$. Let $\overline{X}_\nu^{[n]}$ be the closure of the stratum $X_\nu^{[n]}$ in $X^{[n]}$ and let $\overline{X}_\nu^{(n)}$ be the closure of $X_\nu^{(n)}$ in $X^{(n)}$. It can be proved that $\overline{X}_\nu^{[n]} = \pi^{-1}(\overline{X}_\nu^{(n)})$. If $\nu = 1^{a_1} \cdots n^{a_n}$, then the finite group $\Sigma_\nu := \Sigma_{a_1} \times \cdots \times \Sigma_{a_n}$ acts naturally on $X^{l(\nu)}$, where Σ_{a_i} are the symmetric groups. The quotient X^ν is isomorphic to $X^{(a_1)} \times \cdots \times X^{(a_n)}$. Use the notations in [dCM], we denote $X^{l(\nu)}$ by X^ν . The natural Σ_ν -invariant map $\nu : X^\nu \rightarrow X^{(n)}$ has image $\overline{X}_\nu^{(n)}$. Hence it descends to a map $\nu : X^{(\nu)} \rightarrow X^{(n)}$ which we denote by the same symbol. By using these notations, the correspondences Γ^ν and $\widehat{\Gamma}^\nu$ are defined as follows:

$$\Gamma^\nu := \{(x_1, \dots, x_{l(\nu)}, \mathcal{J}) \in X^\nu \times X^{[n]} : \pi(\mathcal{J}) = \nu_1 x_1 + \cdots + \nu_{l(\nu)} x_{l(\nu)}\} \cong (X^\nu \times_{X^{(n)}} X^{[n]})_{red}.$$

and

$$\widehat{\Gamma}^\nu := \Gamma^\nu / \Sigma_\nu$$

since the correspondence Γ^ν is invariant under the action of Σ_ν on the first factor of the product.

Set $\widehat{\mathcal{X}} = \coprod_{\nu \in \mathfrak{P}(n)} X^{(\nu)}$ and $\widehat{\Gamma} = \coprod_{\nu \in \mathfrak{P}(n)} \widehat{\Gamma}^\nu$. Define integer $m_\nu := (-1)^{n-l(\nu)} \prod_{j=1}^{l(\nu)} \nu_j$. Let

$$\widehat{\Gamma}' := \bigoplus_{\nu \in \mathfrak{P}(n)} \frac{{}^t \widehat{\Gamma}^\nu}{m_\nu}$$

where ${}^t \widehat{\Gamma}^\nu$ is the transposed correspondence of $\widehat{\Gamma}^\nu$. It is proved by de Cataldo and Migliorini that

Theorem 5.2 ([dCM]) *The correspondence $\widehat{\Gamma}$ gives an isomorphism of Chow motives (in the category of finite quotient varieties):*

$$\widehat{\Gamma} = \bigoplus_{\nu \in \mathfrak{P}(n)} \widehat{\Gamma}^\nu : \bigoplus_{\nu \in \mathfrak{P}(n)} (X^{(\nu)}, \Delta_{X^{(\nu)}})(n - l(\nu)) \longrightarrow (X^{[n]}, \Delta_{X^{[n]}}),$$

with the inverse correspondence given by $\widehat{\Gamma}'$.

As a consequence, we obtain the following theorem:

Theorem 5.3 *Let X be a smooth complex projective surface. The natural map of morphic cohomology groups*

$$\widehat{\Gamma}_* = \bigoplus_{\nu \in \mathfrak{P}(n)} \widehat{\Gamma}_*^\nu : \bigoplus_{\nu \in \mathfrak{P}(n)} L^{q-n+l(\nu)} H^{l-2n+2l(\nu)}(X^{(\nu)}, \mathbb{Q}) \longrightarrow L^q H^l(X^{[n]}, \mathbb{Q}) \quad (9)$$

is an isomorphism for any integers $0 \leq l \leq 2q$. (For other l, q both sides becomes zero, therefore the isomorphism holds trivially).

Similarly, we have an isomorphism of Lawson homology groups for all $p, k \in \mathbb{Z}$:

$$\bigoplus_{\nu \in \mathfrak{P}(n)} L_{p-n+l(\nu)} H_{k-2n+2l(\nu)}(X^{(\nu)}, \mathbb{Q}) \longrightarrow L_p H_k(X^{[n]}, \mathbb{Q}) \quad (10)$$

Proof. Note that the $\dim \widehat{\Gamma}^\nu = \dim \Gamma^\nu = n + l(\nu)$. By the functoriality proved in Theorem 4.2, we have $\widehat{\Gamma}_* \widehat{\Gamma}'_* = (\widehat{\Gamma} \widehat{\Gamma}')_* = \text{id}_*$, $\widehat{\Gamma}'_* \widehat{\Gamma}_* = (\widehat{\Gamma}' \widehat{\Gamma})_* = \text{id}_*$. Therefore $\widehat{\Gamma}_*$ gives an isomorphism between morphic cohomology groups by Theorem 5.2, this proves (9). The isomorphism (10) is obtained by applying Theorem 4.3 to Theorem 5.2. \square

Remark 5.3 *A direct proof of the above theorem (without using the language of motives) can be obtained from the method in [dCM] to compute the Chow group of $X^{[n]}$.*

Remark 5.4 *The duality isomorphism between morphic cohomology and Lawson homology can be used to prove the equivalence of the two isomorphisms in Theorem 5.3.*

The above result can be applied to Friedlander-Walker semi-topological K -theory (cf. [FW2] and references therein). Notice that, by [Fu] Corollary 18.3.2, de Cataldo and Migliorini (in [dCM] Theorem 5.4.1) give a decomposition of the rational coefficient Grothendieck group $K_0(X^{[n]})_{\mathbb{Q}} (:= K_0(X^{[n]}) \otimes \mathbb{Q})$:

$$\widehat{\Gamma}_* = \bigoplus_{\nu \in \mathfrak{P}(n)} \widehat{\Gamma}_*^\nu : \bigoplus_{\nu \in \mathfrak{P}(n)} K_0(X^{(\nu)})_{\mathbb{Q}} \xrightarrow{\cong} K_0(X^{[n]})_{\mathbb{Q}} \quad (11)$$

They asked if similar statements hold for higher K -theory.

We do not have an answer for this question. Instead, we give an answer to a similar question for the semi-topological K -theory.

The following is an immediate consequence of Theorem 5.3. It gives a decomposition to $K_*^{sst}(X^{[n]})_{\mathbb{Q}}$ in terms of rational Lawson homology groups.

Corollary 5.4 *There is a natural isomorphism of the semi-topological K -theory groups with rational coefficients*

$$K_p^{sst}(X^{[n]})_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{\nu \in \mathfrak{P}(n)} \bigoplus_j L_j H_{2j+p-n+l(\nu)}(X^{(\nu)})_{\mathbb{Q}}.$$

Proof. It follows from Theorem 4.7 in [FW] and Theorem 5.3. \square

Remark 5.5 *We expect the following isomorphism holds*

$$\bigoplus_{\nu \in \mathfrak{P}(n)} K_{p-n+l(\nu)}^{sst}(X^{(\nu)})_{\mathbb{Q}} \rightarrow K_p^{sst}(X^{[n]})_{\mathbb{Q}}$$

for a smooth projective surface X . However we need a similar isomorphism as that in Theorem 4.7 in [FW] for finite quotient varieties, or more specifically, for $X^{(\nu)}$.

6 Further consequences

6.1 Birational invariants defined by Lawson homology using correspondence

The action of correspondences on Lawson homology gives another proof of the following theorem, which is originally discovered in [Hu] by using diagram chases and blow up formula for Lawson homology together with the Weak Factorization Theorem.

Theorem 6.1 ([Hu]) *If X and Y are birationally equivalent smooth projective complex varieties of dimension n , then $L_1 H_k(X)_{hom} \cong L_1 H_k(Y)_{hom}$ for $k \geq 2$ and $L_{n-2} H_k(X)_{hom} \cong L_{n-2} H_k(Y)_{hom}$ for $k \geq 2(n-2)$.*

Proof. We prove only the isomorphism for $L_1 H_k(-)_{hom}$. The proof of the isomorphism for $L_{n-2} H_k(-)_{hom}$ is similar.

Let Γ be the closure of the graph of a birational map $f : X \dashrightarrow Y$. Note that $\Gamma^t \circ \Gamma$ is the sum of identity correspondence Δ_X and correspondences γ_i 's whose projections are contained in proper subvarieties of X (cf. Example 16.1.11 in [Fu]). Indeed, let $U \subseteq X$, $U' \subseteq Y$ such that f restricts to U is an isomorphism to U' . Let $D_X = X \setminus U$ and $D_Y = Y \setminus U'$. It is easy to see that $(\Gamma^t \circ \Gamma - \Delta_X)$ can be chosen to be supported in $D_X \times D_X$.

By a result of Peters (Lemma 12 in [Pe], cf. Remark 6.1 below), for any $u \in L_1 H_k(X)_{hom}$ and $\gamma \in \mathcal{C}_n(X \times Y)$ such that $p_1(\gamma)$ is a proper subvariety in X , we have $\gamma_*(u) = 0 \in L_1 H_k(Y)_{hom}$. Therefore, $\Gamma_*^t \Gamma_* = (\Gamma^t \Gamma)_* = (\Delta_X)_* = id$ on $L_1 H_k(X)_{hom}$. Symmetrically, $\Gamma_* \Gamma_*^t = id$ on $L_1 H_k(Y)_{hom}$. Therefore Γ_* induces an isomorphism

$$L_1 H_k(X)_{hom} \cong L_1 H_k(Y)_{hom}.$$

\square

Remark 6.1 Lemma 12 in [Pe] asserts that: assume X and Y are smooth projective varieties and $\alpha \subset X \times Y$ is an irreducible cycle of dimension $\dim X = n$, supported in $V \times W$ where $\dim V = v$ and $\dim W = w$. Then $\alpha_* = 0$ if $m < n - v$ or if $m > w$. Moreover, $\alpha_* = 0$ on $L_{n-v}H_*(X)_{hom}$ and on $L_wH_*(X)_{hom}$.

The statement that “ $\alpha_* = 0$ if $m < n - v$ ” is not correct, since $L_pH_k(-)$ is not necessarily zero. But other statements are still valid which we explain here: Let $\tilde{V} \rightarrow V$ and $\tilde{W} \rightarrow W$ be resolutions of singularities. Let $i : \tilde{V} \rightarrow X$ and $j : \tilde{W} \rightarrow Y$ be the natural morphisms, $\tilde{\alpha}$ be the proper transform of α , it can be checked that the following diagram commutes (cf. the proof of Lemma 12 in [Pe]):

$$\begin{array}{ccc}
L_{m-n+v+w}H_{l+2(v+w-n)}(\tilde{V} \times \tilde{W})_{hom} & \xrightarrow{\tilde{\alpha}_*} & L_mH_l(\tilde{V} \times \tilde{W})_{hom} \\
\uparrow p_1^* & & \downarrow (p_2)_* \\
L_{m-n+v}H_{l+2(v-n)}(\tilde{V})_{hom} & & L_mH_l(\tilde{W})_{hom} \\
\uparrow i^* & & \downarrow j_* \\
L_mH_l(X)_{hom} & \xrightarrow{\alpha_*} & L_mH_l(Y)_{hom}
\end{array}$$

As a consequence of this commutative diagram, the conclusion that $\alpha_* = 0$ on $L_{n-v}H_*(X)_{hom}$ is valid by noticing that $L_pH_k(-)_{hom} = 0$ for $p \leq 0$.

6.2 Unirational threefolds and more

In this subsection we describe the Lawson homology for unirational threefolds and fourfolds, and more general the relation between the Lawson homologies of two varieties X and Y .

First of all, we make a remark the motive of a curve. Given a smooth projective curve C and a point $e \in C$, we put $\mathbf{p}_0 = e \times C$ and $\mathbf{p}_2 = C \times e$, then take $\mathbf{p}_1 = \Delta_C - \mathbf{p}_0 - \mathbf{p}_2$ where Δ_C is the diagonal in $C \times C$. Then we have $h(C) = h(pt) \oplus \mathbb{L} \oplus C^+$, where $\mathbb{L} = h(pt)(1)$ is the Lefschetz motive and $C^+ = (C, id - \mathbf{p}_0 - \mathbf{p}_2)$. It is known that the natural map $L_pH_k(C) \rightarrow H_k(C)$, namely the cycle map, is an isomorphism. It is also easy to show that the cycle map commutes with the map Γ_* induced from any correspondence Γ . Therefore $L_pH_k(C^+) \rightarrow H_k(C^+)$ is also an isomorphism.

In §11 of Manin’s paper [M], he gives a motivic decomposition of a unirational threefold X , namely

$$h(X) = h(pt) \oplus a\mathbb{L} \oplus U \otimes \mathbb{L} \oplus a\mathbb{L}^2 \oplus \mathbb{L}^3,$$

where U is a direct summand of a motive of the form $\oplus Y_i^+$, the Y_i being curves. By the argument in the previous paragraph, $L_pH_k(U, \mathbb{Q}) \cong H_k(U, \mathbb{Q})$. Then by Theorem 4.3 (i), a motive decomposition implies the decomposition of rational Lawson homology as well as it is well-known for the rational singular homology, we obtain the following:

Proposition 6.1 *Let X be a three dimensional smooth projective unirational variety over \mathbb{C} . Then the rational Lawson homology group is isomorphic to the corresponding rational singular homology groups, i.e.,*

$$L_p H_k(X, \mathbb{Q}) \cong H_k(X, \mathbb{Q})$$

for any p and k .

By a similar argument, we obtain the following result for a unirational fourfold:

Proposition 6.2 *Let X be a unirational smooth complex projective variety of dimension four. Then the relation of rational Lawson homology and rational singular cohomology is given as follows:*

$$\begin{cases} L_p H_k(X, \mathbb{Q}) \cong H_k(X, \mathbb{Q}), & \text{if } (p, k) \neq (2, 4); \\ L_p H_k(X, \mathbb{Q}) \hookrightarrow H_k(X, \mathbb{Q}) \text{ is injective,} & \text{if } (p, k) = (2, 4). \end{cases}$$

Remark 6.2 *These results can be obtained by the method on the decomposition of diagonals, used by C. Peters in [Pe], since any unirational variety has small Chow groups for zero cycles. Later, M. Voineagu refines Peters' results, which consider only \mathbb{Q} -coefficient Lawson homology groups, to \mathbb{Z} -coefficient in many cases [Vo].*

The above propositions can be generalized to a generically finite rational map as follows:

Proposition 6.3 *If $f : X \dashrightarrow Y$ be a generically finite rational map between smooth projective varieties of dimension n . Then*

$$\dim_{\mathbb{Q}} L_1 H_k(X, \mathbb{Q})_{\text{hom}} \geq \dim_{\mathbb{Q}} L_1 H_k(Y, \mathbb{Q})_{\text{hom}} \quad (12)$$

for $k \geq 2$ and

$$\dim_{\mathbb{Q}} L_{n-2} H_k(X, \mathbb{Q})_{\text{hom}} \geq \dim_{\mathbb{Q}} L_{n-2} H_k(Y, \mathbb{Q})_{\text{hom}} \quad (13)$$

for $k \geq 2(n-2)$. (In case that the right hand side of the inequality has infinite dimension, the left hand side must also be infinite dimensional.)

Proof. Note that $f : X \dashrightarrow Y$ is a rational map of degree $d > 0$. By Hironaka's theorem on the resolution of singularities of mappings, there is a commutative diagram of the form

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow \sigma & \searrow F & \\ X & \dashrightarrow & Y \end{array}$$

where F is a morphism of degree d , and σ is a birational morphism which factors into the composition of blow ups on smooth subvarieties of codimension at least 2. It can be proved that

$$\sigma_* : L_1 H_k(\tilde{X}, \mathbb{Q})_{\text{hom}} \rightarrow L_1 H_k(X, \mathbb{Q})_{\text{hom}} \quad (14)$$

(even $L_1 H_k(\tilde{X})_{\text{hom}} \rightarrow L_1 H_k(X)_{\text{hom}}$) is an isomorphism by reducing the result to one blow up as given in [Hu]. The same is true for codimension two cycles.

It remains to prove that $F_* : L_1 H_k(\tilde{X}, \mathbb{Q})_{\text{hom}} \rightarrow L_1 H_k(Y, \mathbb{Q})_{\text{hom}}$ is surjective. Let $\Gamma_F \in \text{Ch}_n(\tilde{X} \times Y)$ be the graph of F and Γ_F^t be its transpose. Since $F : \tilde{X} \rightarrow Y$ is a morphism of finite degree d between smooth projective varieties, we have $h(\tilde{X}) = h(Y) \oplus (\tilde{X}, \text{id}_{\tilde{X}} - \mathbf{p}, 0)$, where $\mathbf{p} = \frac{1}{d}(\Gamma_F^t) \circ (\Gamma_F)$. Therefore we have $\mathbf{p}_*(L_p H_k(\tilde{X}, \mathbb{Q})) = L_p H_k(Y, \mathbb{Q})$ and $\mathbf{p}_*(H_k(\tilde{X}, \mathbb{Q})) = H_k(Y, \mathbb{Q})$. These two equations imply $\mathbf{p}_*(L_p H_k(\tilde{X}, \mathbb{Q})_{\text{hom}}) = L_p H_k(Y, \mathbb{Q})_{\text{hom}}$ since pull-backs and push-forwards commute with the natural transformation from the Lawson homology to the singular homology. Therefore

$$\dim_{\mathbb{Q}} L_p H_k(\tilde{X}, \mathbb{Q})_{\text{hom}} \geq \dim_{\mathbb{Q}} L_p H_k(Y, \mathbb{Q})_{\text{hom}} \quad (15)$$

From Equations (14) and (15), we get Equation (12). Similar for Equation (13). \square

In particular, for a uniruled threefold X , (recall that a threefold X is uniruled if there is a generic finite map $f : S \times \mathbb{P}^1 \dashrightarrow X$ for some surface S) $L_p H_k(X, \mathbb{Q}) \cong H_k(X, \mathbb{Q})$ if $(p, k) \neq (1, 2)$ or $(2, 4)$ and $L_p H_k(X, \mathbb{Q}) \hookrightarrow H_k(X, \mathbb{Q})$ is injective if $(p, k) = (1, 2)$ or $(2, 4)$.

Remark 6.3 *From the proof of Proposition 6.3, we see that if we have a finite morphism $f : X \rightarrow Y$ between smooth projective varieties, then*

$$\dim_{\mathbb{Q}} L_p H_k(X, \mathbb{Q})_{\text{hom}} \geq \dim_{\mathbb{Q}} L_p H_k(Y, \mathbb{Q})_{\text{hom}}. \quad (16)$$

6.3 Griffiths groups for the product of curves

As the application of the above Proposition 6.3, together results on Griffiths group on generic Jacobian of smooth projective curves, we give examples of products of smooth curves carrying nontrivial Griffiths groups.

Proposition 6.4 *Let C be generic smooth projective curve of genus $g \geq 3$ and let $X = C^g$ be the g -copies of self products of C . Then $\text{Griff}_p(X) \otimes \mathbb{Q}$ are nontrivial for all $1 \leq p \leq g - 2$.*

Proof. Let C be a generic curve of genus $g \geq 3$. Firstly, note that the Jacobian $J(C)$ of C have a non-trivial Griffiths group $\text{Griff}_p(J(C)) \otimes \mathbb{Q}$ for $1 \leq p \leq g - 2$ (cf. [Ce]).

Secondly, it is well known that there is a birational morphism from the g -th symmetric product $C^{(g)}$ of C to $J(C)$, i.e., $\sigma : C^{(g)} \rightarrow J(C)$ is a birational morphism. Therefore,

$$\dim_{\mathbb{Q}} \{\text{Griff}_p(C^{(g)}) \otimes \mathbb{Q}\} \geq \dim_{\mathbb{Q}} \{\text{Griff}_p(J(C)) \otimes \mathbb{Q}\}$$

by the proof to Equation (16) in Remark 6.3. For the special cases $p = 1$ or $g - 2$, $\text{Griff}_1(C^{(g)}) \cong \text{Griff}_1(J(C))$ and $\text{Griff}_{g-2}(C^{(g)}) \cong \text{Griff}_{g-2}(J(C))$ also follows from Theorem 6.1.

Finally, since the natural projection $\pi : C^g \rightarrow C^{(g)}$ is of finite degree $g!$, we have

$$\dim_{\mathbb{Q}}\{\text{Griff}_p(C^g) \otimes \mathbb{Q}\} \geq \dim_{\mathbb{Q}}\{\text{Griff}_p(C^{(g)}) \otimes \mathbb{Q}\}$$

from Remark 6.3.

The combination these statements completes the proof of the proposition. \square

Remark 6.4 *Since all Griffiths groups for curves are zero and $L_1H_2(X)_{\text{hom}} \cong \text{Griff}_1(X)$ for a smooth projective variety X , we obtain that the Künneth type formula in general can **not** hold for Griffiths groups and Lawson homology.*

Remark 6.5 *It was constructed explicitly by B. Harris in [Hb] for the Fermat curve C of degree 4 (hence $g(C) = 3$) with $\text{Griff}_1(J(C)) \neq 0$. In fact, $\text{Griff}_1(J(C)) \otimes \mathbb{Q} \neq 0$. By the proof in Proposition 6.4, we get $\text{Griff}_1(X) \otimes \mathbb{Q} \neq 0$ for $X = C^3$ the 3 times self product of C .*

7 Appendix

We would like to make a remark on birational morphisms.

Proposition 7.1 *Let $f : X \rightarrow Y$ be a morphism between smooth complex projective n -dimensional varieties. Suppose that $f_* : \text{Ch}_p(X) \rightarrow \text{Ch}_p(Y)$ is isomorphic for $p = n - 1$ and $p = n$. Then f is an isomorphism.*

Proof. The isomorphism of f_* for $p = n$ implies that f is a birational morphism. Indeed, $\text{Ch}_n(X) \cong \text{Ch}_n(Y) \cong \mathbb{Z}$, and f_* is an multiplication by d where d is the degree of f , hence $d = 1$. Then we apply the fact that, if a birational morphism f is not an isomorphism, then there is an exceptional subvariety $Z \subset X$, i.e. $\text{codim } Z = 1$ and $\text{codim } f(Z) \geq 2$. It is easy to show that $[Z] \neq 0 \in \text{Ch}_{n-1}(X)$ but $f_*([Z]) = 0 \in \text{Ch}_{n-1}(Y)$, contradicts to the fact that f_* is an isomorphism. \square

Remark 7.1 *The statement is amazing by comparing to the corresponding one between topological manifolds: For a continuous map $F : M \rightarrow N$ between oriented topological manifolds, even if $F_* : H_k(M, \mathbb{Z}) \rightarrow H_k(N, \mathbb{Z})$ are isomorphisms for all k , we don't know whether $F : M \rightarrow N$ is homeomorphic.*

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Wenchuan Hu, Department of Mathematics, MIT, Room 2-101, 77 Massachusetts Avenue
Cambridge, MA 02139 Email: wenchuan@math.mit.edu

Li Li, Department of Mathematics, Univ. of Illinois, Urbana, IL 61801
Email: llpku@math.uiuc.edu